Finite and Infinite Calculus & Infinite Sums

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Overview

Mathematicians have developed a “finite calculus” analogous to the more traditional infinite calculus, by which it is possible to approach summation in a nice, systematic fashion.

Infinite calculus is based on the properties of the **derivative operator** D, defined by

$$Df(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Finite calculus is based on the properties of the **difference operator** Δ, defined by

$$\Delta f(x) = f(x+1) - f(x).$$

The symbols D and Δ are called **operators** because they operate on functions to give new functions ; they are functions of functions that produce functions.

Let $f(x) = x^m$. Then $Df(x) = mx^{m-1}$. But Δ does not produce an equally elegant result. For example,

$$\Delta(x^3) = 3x^2 + 3x + 1 \neq 3x^2.$$

But there is a type of "mth power" that does transform nicely under Δ, and this is what makes finite calculus interesting. Such **mth powers (factorial functions)** are defined by the rule

$$x^{\underline{m}} = x(x-1)\cdots(x-m+1), \quad \text{for } m \geq 0.$$

There is also a corresponding definitions where the factors go up and up:

$$x^{\overline{m}} = x(x+1)\cdots(x+m-1), \quad \text{for } m \geq 0.$$

When $m = 0$, we have $x^{\underline{0}} = x^{\overline{0}} = 1$, because a product of no factors is **conventionally taken** to be 1 (just as a sum of no terms is conventionally 0.)

The quantity $x^{\underline{m}}$ is called "x to the m falling", similarly, $x^{\overline{m}}$ is "x to the m rising".

These functions are also called **falling factorial powers** or rising factorial powers, since they are closely related to the factorial function

$$n! = n(n-1)\cdots 1.$$

In fact,

$$n! = n^{\underline{n}} = 1^{\overline{n}}.$$

We defined $x^{\underline{m}}$ for $m \geq 0$.

To get from $x^{\underline{3}}$ to $x^{\underline{2}}$, we divide by $x-2$. That is, $x^{\underline{2}} = \frac{x^{\underline{3}}}{x-2}$.

To get from $x^{\underline{2}}$ to $x^{\underline{1}}$, we divide by $x-1$. That is, $x^{\underline{1}} = \frac{x^{\underline{2}}}{x-1}$.

To get from $x^{\underline{1}}$ to $x^{\underline{0}}$, we divide by x. That is, $x^{\underline{0}} = \frac{x^{\underline{1}}}{x}$.

To go from $x^{\underline{0}}$ to $x^{\underline{-1}}$, we should divide by $x+1$. Hence $x^{\underline{-1}} = \frac{x^0}{x+1}$.
Similarly, $x^{\underline{-m}} = \frac{1}{(x+1)(x+2)\cdots(x+m)}$ for $m > 0$.
We shall later define falling powers for real or even complex number m.

Exercise

1. *Prove that the formula $x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}$ (falling power version) for falling powers (analogous to the law of exponents, $x^{m+n} = x^m + x^n$ for ordinary powers).*

Let m be an integer. Verify that

$$\Delta x^m = m x^{\underline{m-1}} \quad \textit{when } m < 0.$$

Falling powers $x^{\underline{m}}$ are especially nice with respect to Δ.

$$\Delta(x^{\underline{m}}) = m x^{\underline{m-1}},$$

hence the finite calculus has a handy low to match $D(x^m) = m x^{m-1}$.

The operator D of inifinite calculus has an inverse, the anti-derivative (or integration) operator $\int$. The **Fundamental Theorem of Calculus** relates D to $\int$:

$$g(x) = Df(x) \iff \int g(x)dx = f(x) + c.$$

Here $\int g(x)dx$, the **indefinite integral** of $g(x)$, is the class of functions whose derivative is $g(x)$ and c for "indefinite integrals" is an arbitrary constant.

Analogously, Δ has an inverse, the **anti-difference** (or **summation**) **operator** $\sum$; and there is another fundamental theorem :

$$g(x) = \Delta f(x) \iff \sum g(x)\delta x = f(x) + c.$$

Here $\sum g(x)\delta x$, the **indefinite sum** of $g(x)$, is the class of functions whose difference is $g(x)$ and c for "indefinite sums" is an arbitrary function $p(x)$ such that $p(x+1) = p(x)$.

Exercise

2. *Find $\Delta f(x)$, where $f(x)$ is the periodic function $a + b\sin 2\pi x$.*

Infinite calculus has definite integrals: If $g(x) = Df(x)$, then

$$\int_a^b g(x)dx = f(x)\Big|_a^b = f(b) - f(a).$$

Finite calculus has definite sums: If $g(x) = \Delta f(x)$, then

$$\sum_a^b g(x)\delta x = f(x)\Big|_a^b = f(b) - f(a).$$

Assume that $g(x) = \Delta f(x) = f(x+1) - f(x)$.

Special cases:

- If $a = b$, we have $\sum_a^b g(x)\delta x = f(b) - f(a) = 0$.
- Next, if $b = a + 1$, the result is

$$\sum_a^{a+1} g(x)\delta x = f(a+1) - f(a) = g(a).$$

- More generally, if b increases by 1, we have the difference

$$\begin{aligned}\sum_a^{b+1} g(x)\delta x - \sum_a^{b} g(x)\delta x &= [f(b+1) - f(a)] - [f(b) - f(a)] \\ &= f(b+1) - f(b) = g(b).\end{aligned}$$

- When a and b are integers with $b \geq a$,

$$\sum_a^b g(x)\delta x = \sum_{k=0}^{b-1} g(k) = \sum_{a \leq k < b} g(k).$$

In other words, the definite sums is the same as an ordinary sum with limits, but excluding the value at the upper limit.

- What happens when $b < a$?

$$\sum_a^b g(x)\delta x f(b) - f(a) = -(f(a) - f(b)) = -\sum_a^b g(x)\delta x.$$

- For any integers a, b, c

$$\sum_a^b g(x)\delta x + \sum_b^c g(x)\delta x = \sum_a^c g(x)\delta x.$$

Suppose we want to find the sum of the form

$$\sum_{a \le k < b} g(k) = \sum_{a}^{b} g(x)\delta x.$$

If we are able to find an **anti-difference** function f such that

$$g(x) = f(x+1) - f(x),$$

then

$$\begin{aligned} \sum_{a \le k < b} g(k) &= \sum_{a \le k < b} f(x+1) - f(x) \quad \text{(telescoping series)} \\ &= [f(a+1) - f(a)] + [f(a+2) - f(a+1)] + \cdots + [f(b) - f(b-1)] \\ &= f(b) - f(a). \end{aligned}$$

We shall see that definite summation gives us a simple way to compute sums of falling powers.

Ordinary powers can also be summed in new way, if we express them in terms of falling powers. For example,

$$k^2 = k^{\underline{2}} + k^{\underline{1}}$$

hence

$$\sum_{0 \le k < n} k^2 = \frac{k^{\underline{3}}}{3} + \frac{k^{\underline{2}}}{2}\bigg|_{k=0}^{k=n} = \frac{1}{3}n\left(n - \frac{1}{2}\right)(n-1).$$

Replacing n by $n+1$ gives us yet another way to compute the value of

$$\Box_n = \sum_{0 \le k \le n} k^2$$

in closed form.

It is always possible to convert between ordinary powers and factorial powers by using Stirling numbers, which will be later studied.

Falling powers are very nice for sums.

Exercises

3. *Prove that* $(x+y)^{\underline{2}} = x^{\underline{2}} + 2x^{\underline{1}}y^{\underline{1}} + y^{\underline{2}}$.
4. *State and prove that the factorial binomial theorem.*
5. *Prove that the summation property* $\sum_a^b x^{\underline{m}} \delta x = \frac{x^{\underline{m+1}}}{m+1}\Big|_a^b$ *holds for any integer* $m \neq -1$ *and any* x.
6. *Does the summation property hold for* $m = -1$*?*

Recall that for integration we use

$$\int_a^b x^{\underline{-1}} dx = \log x \Big|_a^b$$

when $m = -1$.

What is a finite analog of $\log x$?

What is the function $f(x)$ satisfying

$$x^{\underline{1}} = \frac{1}{x+1} = \Delta f(x) = f(x+1) - f(x).$$

Hence $f(x)$ is the harmonic number

$$H_x = 1 + \frac{1}{2} + \cdots + \frac{1}{x}.$$

Thus H_x is the discrete analog of the continuous $\log x$.

We shall define H_x for noninteger x and for large values of x, the value of $H_x - \log x$ is approximately

$$0.577 + \frac{1}{2x}.$$

Hence H_x and $\log x$ are not only analogous, their values usually differ by less than 1.

We can now give a complete description of the sums of falling powers:

$$\sum_a^b x^{\underline{m}} \delta x = \frac{x^{\underline{m+1}}}{m+1}\bigg|_a^b \quad if\, m \neq -1$$

$H_x\Big|_b^a$ if $m = -1$.

This formula indicates why harmonic numbers tend to pop up in the solutions to dicrete problems like the analysis of quicksort, just as so-called natural logarithms aris naturally in the solutions to continuous problems.

Exercise

7. *Prove that* 2^x *is the discrete analog for* e^x, *called the* discrete exponential function.

Despite all the parallels betweeen continuous and discrete math, some continuous notions have no discrete analog.

For example, the **chain rule** for the derivative of a function of a function ; but there is no corresponding chain rule of finite calculus, because there is no nice form for $\Delta f(g(x))$.

Discrete change-of-variables is hard, except in certain cases like the replacement of x by $c \pm x$.

However, $\Delta(f(x)g(x))$ does have a fairly nice form, and it provides us with a reule for **summation by parts**, the finite analog of what infinite calculus calls integration by parts.

Let us recall the formula

$$D(uv) = uDv + vDu$$

of infinite calculus leads to the rule for integration by parts,

$$\int uDv = uv - \int vDu,$$

after integration and rearranging terms; we can do a similar thing in finite calculus.

We start by applying the difference operator to the product of two functions $u(x)$ and $v(x)$:

$$\begin{aligned} \Delta(u(x)v(x)) &= u(x+1)v(x+1) - u(x)v(x) \\ &= u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x) \\ &= u(x)\Delta v(x) + v(x+1) + v(x+1)\Delta u(x). \end{aligned}$$

This formula can be put into a convenient form using the **shift operator** E, defined by

$$Ef(x) = f(x+1).$$

Hence $\Delta(uv) = u\Delta v + Ev\Delta u$.

Taking the indefinite sum on both sides of this equation, and rearranging its terms, yields the rule for summation by parts:

$$\sum u\Delta v = uv - \sum Ev\Delta u.$$

This rule is useful when the sum on the left is harder to evaluate than the one on the right.

Exercise

8. *Find the sum of the following :*

- $\displaystyle\sum_{k=0}^{n} k2^k$
- $\displaystyle\sum_{0\le k<n} kH_k$.

Infinite Sums

- The methods we have used for manipulating $\sum$'s are **not** always valid when infinite sums are involved.
- There is a large, easily understood class of infinite sums for which all the operations we have been performing are perfectly legitimate.

Suppose all the terms a_k are non-negative. If there is a **bounding constant** A such that

$$\sum_{k \in F} a_k \leq A$$

for all finite subsets F of K, then we define $\sum_{k \in K}$ to be the least such A.

It follows from a well-known properties of the real numbers that the set of all such A always contains a smallest element.

The definition has been formulated carefully so that it doesn't depend on any order that might exist in the index set K

But there is **no bounding constant** A, we say that

$$\sum_{k \in K} a_k = \infty.$$

If K is the set of non-negative integers, then for non-negative terms a_k, we have $\sum_{k \geq 0} a_k = \lim_{n \to \infty} \sum_{k=0}^{n} a_k$.

- $\sum_{k \geq 0} x^k$ can be calculated as follows:

$$\sum_{k \geq 0} x^k = \lim_{n \to \infty} \frac{1 - x^{n+1}}{1 - x} = \begin{cases} \frac{1}{1-x} & \text{if } 0 \leq x < 1 \\ \infty & \text{if } x \geq 1. \end{cases}$$

- $$\sum_{k\geq 0} \frac{1}{(k+1)(k+2)} = \lim_{n\to\infty} \sum_{k=0}^{n} k^{\underline{-2}} = \lim_{n\to\infty} \frac{k^{\underline{-1}}}{-1}\Big]_0^n = 1.$$
- There is something **flaky** about a sum that gives different values when its terms are added up in different ways.
- How to find $\sum_{k\in K} a_k$, where a_k is a real-valued term defined for each $k \in K$?
- Any real number x can be written as the difference of its positive and negative parts, $x = x^+ - x^-$, where $x^+ = x.[x > 0]$ and $x^- = -x.[x < 0]$. Because a_k^- and a_k^- are non-negative, we can find value for the infinite sums $\sum_{k\in K} a_k^+$ and $\sum_{k\in K} a_k^-$.

  Hence $\sum_{k\in K} = a_k = \sum_{k\in K} a_k^+ - \sum_{k\in K} a_k^-$ unless the right-hand sums are both equal to ∞.

Let $A^+ = \sum_{k \in K} a_k^+$ and $A^- = \sum_{k \in K} a_k^-$.

- If A^+ and A^- are both finite, the sum $\sum_{k \in K} a_k$ is said to **converge absolutely** to the value $A = A^+ - A^-$.
- If $A^+ = \infty$ but A^- is finite, the sum $\sum_{k \in K} a_k$ is said to **diverge** to $+\infty$.
- If $A^- = \infty$ but A^+ is finite, the sum $\sum_{k \in K} a_k$ is said to **diverge** to $-\infty$.
- If $A^+ = A^- = \infty$, we call $\sum_{k \in K} a_k$ is undefined.

Exercise

9. *We started with a definition that worked for non-negative terms, then we extend it to real-valued terms. Extend the definition if the terms of complex numbers.*

All of the manipulations we have done for finite sums are perfectly valid whenever we are dealing with "**sums that converge absolutely**".

Each of the following transformation rules preserves the value of all absolutely convergent sums.

- distributive law
- commutative law
- associative law
- rule for summing first on one index variable.

Absolutely convergent sums over 2 or more indices can always be summed first with respect to any one of those indicies.

References

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